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Asymptotic Behavior of Solutions of a Second Order Nonlinear Differential Equation

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1. INTRODUCTION

In this paper we consider the differential equation

$$(a(t)\,x')' + q(t)\,f(x)\,g(x') = r(t) \tag{1}$$

where a, q, $r : [t_0, \infty) \to R$, $f, g : R \to R$, $a(t) > 0$, $q(t) > 0$, $g(x') > 0$, and a, q, r, f, and g are continuous. Equations of this type have been studied by many authors and an excellent survey of known results can be found in Ref. [16]. More recent contributions include Refs. [1–8, 10–14, 18]. The approach we take in studying the asymptotic behavior of solutions of Eq. (1) is new. By examining the quotient $r(t)/q(t)$ as $t \to \infty$ we are able to obtain boundedness and other behavioral results without requiring that the forcing term, $r(t)$, be "small" in some sense. In fact, our results will allow $r(t)$ to become unbounded as $t \to \infty$. This is a significant departure from previous studies of Eq. (1). In addition, we will be able to relax some of the conditions that other authors place on the functions in Eq. (1). Finally, in our discussion of the asymptotic behavior of solutions of Eq. (1), we will include a class of solutions, the Z-type solutions (see Section 3 for the definition), not considered previously.

First, we begin with some new continuability and boundedness results.

2. CONTINUABILITY AND BOUNDEDNESS

We will write Eq. (1) as the system

$$\begin{aligned} x' &= y, \\ y' &= (-a'(t)y - q(t)\,f(x)\,g(y) + r(t))/a(t). \end{aligned} \tag{2}$$

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Let $q'(t)_+ = \max\{q'(t), 0\}$ and $q'(t)_- = \max\{-q'(t), 0\}$ so that we have $q'(t) = q'(t)_+ - q'(t)_-$. A similar decomposition holds for $a(t)$. Define

$$F(x) = \int_0^x f(s)\,ds, \qquad G(y) = \int_0^y [s/g(s)]\,ds,$$

$$p(t) = \exp\left(-\int_{t_0}^t [q'(s)_-/q(s)]\,ds\right),$$

and

$$b(t) = \exp\left(-\int_{t_0}^t [a'(s)_-/a(s)]\,ds\right).$$

Assume that there exist nonnegative constants m and n such that

$$|\,y\,|/g(y) \leqslant m + nG(y). \tag{3}$$

THEOREM 1. *Suppose* (3) *holds,* $a'(t) \geqslant 0$, $F(x)$ *is bounded from below, and* $G(y) \to \infty$ *as* $|\,y\,| \to \infty$. *Then all solutions of* (2) *can be defined for all* $t \geqslant t_0$.

Proof. Suppose there is a solution $(x(t), y(t))$ of (2) and $T > t_0$ such that $\lim_{t\to T^-} [|\,x(t)| + |\,y(t)|] = +\infty$. There exists $K > 0$ such that $F(x) \geqslant -K$ for all x. Define $V(x, y, t) = p(t)[(F(x) + K)/a(t) + G(y)/q(t)]$. Then $V' = p(t)\{-(F(x) + K)a'(t)/a^2(t) + f(x)x'/a(t) - G(y)q'(t)/q^2(t) + yy'/g(y)q(t) - (F(x) + K)q'(t)_-/a(t)q(t) - G(y)q'(t)_-/q^2(t)\} \leqslant p(t)\{-G(y)[q'(t) + q'(t)_-]/q^2(t) + r(t)\,y/g(y)\,q(t)\,a(t)\} \leqslant p(t)\,r(t)\,y/g(y)\,q(t)\,a(t)$. Integrating, we obtain

$$V(t) \leqslant V(t_0) + \int_{t_0}^t [p(s)\,r(s)\,y(s)/g(y(s))\,q(s)\,a(s)]\,ds.$$

By (3) and the fact that $p(t)\,G(y(t))/q(t) \leqslant V(t)$, we have

$$p(t)\,|\,y(t)|/g(y(t))\,q(t) \leqslant mp(t)/q(t) + nV(t_0) + n\int_{t_0}^t [p(s)\,r(s)\,y(s)/g(y(s))\,q(s)\,a(s)]\,ds.$$

On $[t_0, T]$, $p(t)/q(t)$ is bounded so we have

$$p(t)\,|\,y(t)|/g(y(t))\,q(t) \leqslant K_1 + n\int_{t_0}^t [p(s)\,|\,r(s)|\,|\,y(s)|/g(y(s))\,q(s)\,a(s)]\,ds$$

for some $K_1 > 0$, and an application of Gronwall's inequality yields

$$p(t)\,|\,y(t)|/g(y(t))\,q(t) \leqslant K_1 \exp\left(n\int_{t_0}^{t} [|\,r(s)|/a(s)]\,ds\right)$$

$$\leqslant K_1 \exp\left(n\int_{t_0}^{t} [|\,r(s)|/a(s)]\,ds\right) \leqslant K_2 < \infty.$$

Hence

$$V(t) \leqslant V(t_0) + \int_{t_0}^{t} K_2[|\,r(s)|/a(s)]\,ds \leqslant K_3 < \infty.$$

Thus $p(t)\,G(y(t))/q(t) \leqslant K_3$ on $[t_0, T)$ and so $G(y(t))$ is bounded on $[t_0, T)$. This implies that $y(t) = x'(t)$ is bounded on $[t_0, T)$, and an integration yields that $x(t)$ is also bounded on $[t_0, T)$ contradicting the assumption that $(x(t), y(t))$ was a solution of (2) with finite escape time.

The requirement that $a'(t) \geqslant 0$ in Theorem 1 can be dropped by replacing (3) with a stronger condition. We then obtain the following extension of a result of Burton and Grimmer [2].

THEOREM 2. *Assume that $F(x)$ is bounded from below, $G(y) \to \infty$ as $|\,y\,| \to \infty$, and there are positive constants M and k such that*

$$y^2/g(y) \leqslant MG(y) \text{ for } |\,y\,| \geqslant k. \tag{4}$$

Then all solutions of (2) *can be defined for all $t \geqslant t_0$.*

Proof. Let $(x(t), y(t))$ be a solution of (2) such that $|\,x(t)| + |\,y(t)| \to \infty$ as $t \to T^-$, and let $-K < 0$ be a lower bound for $F(x)$. Define $V(x, y, t) = b(t)p(t)[(F(x) + K)/a(t) + G(y)/q(t)]$; then $V' = b(t)p(t)\{-(F(x) + K)\,a'(t)/a^2(t) + f(x)\,y/a(t) - G(y)\,q'(t)/q^2(t) - a'(t)\,y^2/g(y)\,q(t)\,a(t) - f(x)\,y/a(t) + r(t)\,y/g(y)\,q(t)\,a(t) - [(F(x) + K)/a(t) + G(y)/q(t)](a'(t)_-/a(t) + q'(t)_-/q(t))\} \leqslant b(t)p(t)\{-a'(t)y^2/g(y)q(t)a(t) + r(t)y/g(y)q(t)a(t)\}$. If $|\,y\,| \leqslant \max\{k, 1\}$, $y^2/g(y) \leqslant D$ for some $D > 0$, so $y^2/g(y) \leqslant D + MG(y)$ for all y. Also, for $|\,y\,| \leqslant \max\{k, 1\}$, $|\,y\,|/g(y) \leqslant D_1$, $D_1 > 0$, and for $|\,y\,| \geqslant \max\{k, 1\}$, $|\,y\,|/g(y) \leqslant y^2/g(y)$ so $|\,y\,|/g(y) \leqslant D_1 + y^2/g(y) \leqslant D_1 + D + MG(y)$ for all y. Integrating V' we then have

$$b(t)\,p(t)\,G(y(t))/q(t)$$

$$\leqslant V(t_0) + \int_{t_0}^{t} M\{b(s)\,p(s)\,[|\,a'(s)| + |\,r(s)|]\,G(y(s))/q(s)\,a(s)\}\,ds$$

$$+ \int_{t_0}^{t} \{b(s)\,p(s)\,[D\,|\,a'(s)| + (D_1 + D)\,|\,r(s)|]/q(s)\,a(s)\}\,ds.$$

Now the last integral is bounded on $[t_0, T]$, so

$$b(t)\,p(t)\,G(y(t))/q(t) \leqslant K_1 \exp \int_{t_0}^{t} \{M[|\,a'(s)| + |\,r(s)|]/a(s)\}\,ds,$$

for some $K_1 > 0$. Hence $G(y(t))$ is bounded on $[t_0, T)$ and the remainder of the proof follows as in the previous theorem.

Remark. In the preceding continuability theorems as well as in the following boundedness results we have not required, as do most authors, the condition $xf(x) > 0$ if $x \neq 0$, or even that $F(x) \geqslant 0$.

THEOREM 3. *Suppose Eq.* (3) *holds,* $a'(t) \geqslant 0, F(x) \to \infty$ *as* $|\,x\,| \to \infty$, $\int_{t_0}^{\infty} |r(s)|\,ds < \infty$, $a(t)$ *is bounded, and*

$$\int_{t_0}^{\infty} [q'(s)_-/q(s)]\,ds < \infty. \tag{5}$$

Then all solutions of Eq. (1) *are bounded.*

Proof. Since $F(x) \to \infty$ as $|\,x\,| \to \infty$, $F(x)$ is bounded from below, say $F(x) \geqslant -K$ for some $K > 0$. Defining V as in the proof of Theorem 1, differentiating and integrating we obtain

$$p(t)\,|\,y(t)|/g(y(t))\,q(t) \leqslant mp(t)/q(t) + nV(t_0)$$
$$+\, n \int_{t_0}^{t} [p(s)\,r(s)\,y(s)/g(y(s))\,q(s)\,a(s)]\,ds.$$

Now $p(t) \leqslant 1$ and (5) bounds $q(t)$ away from zero so $mp(t)/q(t) + nV(t_0) \leqslant K_1$. Then by Gronwall's inequality

$$p(t)\,|\,y(t)|/g(y(t))\,q(t) \leqslant K_1 \exp \left[(n/a(t_0)) \int_{t_0}^{\infty} |\,r(s)|\,ds\right].$$

As in the proof of Theorem 1, it follows that $V(t)$ is bounded so $p(t)F(x(t))/a(t)$ is bounded and hence $x(t)$ is bounded.

COROLLARY 4. If, in addition to the hypotheses of the Theorem, we have $q(t)$ bounded from above and $G(y) \to \infty$ as $|\,y\,| \to \infty$, then all solutions of system (2) are bounded.

Proof. Since $V(t)$ is bounded, $p(t)\,G(y(t))/q(t)$ is bounded, and the conclusion follows.

Corollary 4 extends Corollary 4 in Ref. [3] and Theorem 6 in Ref. [2].

Remark. If, in Theorem 1 or 3, $r(t) \equiv 0$, then condition (3) can be dropped.

Our next boundedness result is patterened after Theorem 2. It also extends the above mentioned results in Refs. [2] and [3].

THEOREM 5. *Let conditions* (4) *and* (5) *hold,* $F(x) \to \infty$ *as* $|x| \to \infty$,

$$\int_{t_0}^{\infty} |r(s)|\, ds < \infty,$$

and

$$\int_{t_0}^{\infty} [a'(s)_-/a(s)]\, ds < \infty \qquad \text{and} \qquad a(t) \leqslant a_2\,. \tag{6}$$

Then all solutions of Eq. (1) *are bounded.*

Proof. Define V as in the proof of Theorem 2. Then

$$\begin{aligned} V' &\leqslant b(t)\,p(t)\{-a'(t)\,y^2/g(y)\,q(t)\,a(t) + r(t)\,y/g(y)\,q(t)\,a(t)\} \\ &\leqslant b(t)\,p(t)\{y^2 a'(t)_-/g(y)\,q(t)\,a(t) + r(t)\,y/g(y)\,q(t)\,a(t)\}. \end{aligned}$$

If $|y| \leqslant 1$, then $|y|/g(y) \leqslant B$ for some $B > 0$, so $|y|/g(y) \leqslant B + y^2/g(y)$ for all y. Thus integrating V' we obtain

$$\begin{aligned} V(t) &\leqslant V(t_0) \\ &\quad + \int_{t_0}^{t} \{b(s)\,p(s)\,[a'(s)_-/a(s) + |r(s)|/a(s)]\,y^2(s)/g(y(s))\,q(s)\}\, ds \\ &\quad + B \int_{t_0}^{t} [|r(s)|/q(s)\,a(s)]\, ds. \end{aligned}$$

Notice that the second integral above is bounded for all $t \geqslant t_0$. Now $y^2/g(y) \leqslant D$ for $|y| \leqslant k$ so by (4) we have

$$\begin{aligned} & b(t)\,p(t)\,y^2(t)/g(y(t))\,q(t) \\ &\quad \leqslant b(t)\,p(t)\,D/q(t) + b(t)\,p(t)\,MG(y(t))/q(t) \leqslant K_1 \\ &\qquad + M \int_{t_0}^{t} \{b(s)\,p(s)\,[a'(s)_-/a(s) + |r(s)|/a(s)]\,y^2(s)/g(y(s))\,q(s)\}\, ds, \end{aligned}$$

and an application of Gronwall's inequality yields

$$b(t)\,p(t)\,y^2(t)/g(y(t))\,q(t) \leqslant K_1 \exp\left(M \int_{t_0}^{t} [a'(s)_-/a(s) + |r(s)|/a(s)]\, ds\right) \leqslant K_2$$

for all $t \geqslant t_0$.

Hence

$$V(t) \leqslant V(t_0) + K_2 \int_{t_0}^{t} [a'(s)_-/a(s) + |r(s)|/a(s)]\, ds$$

so $V(t)$ is bounded. By (5) and (6) we have that $F(x(t))$ is bounded and thus $x(t)$ is bounded.

COROLLARY 6. If, in addition to the hypotheses of Theorem 5, we have $q(t)$ bounded and $G(y) \to \infty$ as $|y| \to \infty$, then all solutions of (2) are bounded.

3. Asymptotic Behavior

In this section we discuss the asymptotic behavior of solutions of Eq. (1). Several of our results are obtained by using the quotient $H(t) = r(t)/q(t)$. Other authors, for example, Chang [3], Lalli [9], Wong [15], and Zarghamee and Mehri [17], have utilized the quotient $a(t)/q(t)$. The present authors [5, 6] obtained some results for a less general equation in case $H(t)$ is monotonic.

The previous section contained sufficient conditions for solutions of Eq. (1) to be continuable. Without further mention, we note that the results in this section pertain only to the continuable solutions of Eq. (1). Obviously, by combining results in this section with Theorems 1 or 2, this provision would not be necessary. The following classification of solutions will be used.

DEFINITION. A solution $x(t)$ of Eq. (1) will be called nonoscillatory if there exists $t_1 \geqslant t_0$ such that $x(t) \neq 0$ for $t \geqslant t_1$; the solution will be called oscillatory if for any given $t_1 \geqslant t_0$ there exist t_2 and t_3 satisfying $t_1 < t_2 < t_3$, $x(t_2) > 0$, and $x(t_3) < 0$; and it will be called a Z-type solution if it has arbitrarily large zeros but is ultimately nonnegative or nonpositive.

The following conditions will often be required:

$$\int_{t_0}^{\infty} q(s)\, ds = \infty, \tag{7}$$

$$xf(x) > 0 \text{ if } x \neq 0, \tag{8}$$

$$f'(x) \geqslant 0 \text{ for all } x, \tag{9}$$

$$g(x') \geqslant c > 0, \tag{10}$$

$$\int_{t_0}^{\infty} [1/a(s)]\, ds = \infty. \tag{11}$$

THEOREM 7. *Suppose* $a(t) \geqslant a_1 > 0$, $q(t) \geqslant q_1 > 0$, $\int_{t_0}^{\infty} r(s)\, ds$ *is bounded,* *and conditions* (8)–(11) *hold. If* $x(t)$ *is a nonoscillatory or Z-type solution of Eq.* (1), *then* $\lim_{t\to\infty} x(t) = 0$.

Proof. We will write Eq. (1) as the system

$$x' = \left(y + \int_{t_0}^{t} r(s)\, ds\right)\Big/ a(t), \qquad y' = -q(t) f(x)\, g(x'). \tag{2$'$}$$

Let $x(t)$ be a nonoscillatory solution of (1), say $x(t) > 0$ for $t \geqslant t_1 \geqslant t_0$. Suppose $\liminf_{t\to\infty} x(t) \neq 0$. Then there exists $t_2 \geqslant t_1$ such that $x(t)$ is bounded from below for $t \geqslant t_2$. This, together with (9), implies that $f(x(t)) \geqslant A > 0$ for $t \geqslant t_2$. Thus

$$y(t) - y(t_2) = -\int_{t_2}^{t} q(s) f(x(s))\, g(x'(s))\, ds \leqslant -cA \int_{t_2}^{t} q(s)\, ds \to -\infty$$

as $t \to \infty$. We then have

$$x'(t) = \left(y(t) + \int_{t_0}^{t} r(s)\, ds\right)\Big/ a(t) \leqslant -1/a(t)$$

for $t \geqslant t_3$, for some $t_3 \geqslant t_2$. Integrating, we obtain that

$$x(t) \leqslant x(t_3) - \int_{t_3}^{t} [1/a(s)]\, ds \to -\infty$$

as $t \to \infty$, contradicting the fact that $x(t) > 0$ for $t \geqslant t_1$.

Next, we have $y'(t) = -q(t) f(x(t))\, g(x'(t)) \leqslant 0$ so $y(t)$ is bounded from above and thus $x'(t) \leqslant B/a_1$ for some $B > 0$. If $\limsup_{t\to\infty} x(t) = D > 0$, then there exist sequences $\{w_n\}$ and $\{s_n\}$ with $w_n < s_n < w_{n+1}$, $x(w_n) = D/3$, $x(s_n) = 2D/3$, and $D/3 \leqslant x(t) \leqslant 2D/3$ for $w_n \leqslant t \leqslant s_n$. Since $x'(t) \leqslant B/a_1$,

$$\begin{aligned} D/3 = x(s_n) - x(w_n) &= \int_{w_n}^{s_n} x'(s)\, ds \\ &\leqslant \int_{w_n}^{s_n} (B/a_1)\, ds = B(s_n - w_n)/a_1, \end{aligned}$$

so $s_n - w_n \geqslant a_1 D/3B$. Choose $K > 0$ such that $f(x(t)) \geqslant K$ for $x(t) \geqslant D/3$. Then

$$\int_{w_1}^{\infty} q(s) f(x(s))\, g(x'(s))\, ds \geqslant \sum_{n=1}^{\infty} K q_1 c a_1 D/3B$$

which is a contradiction since, as we see from the first part of the proof, the integral on the left must converge.

If $x(t)$ is a nonnegative Z-type solution, then $\liminf_{t\to\infty} x(t) = 0$, and the preceding argument shows that $\limsup_{t\to\infty} x(t)$ is also zero. The proof in case $x(t) \leqslant 0$ is similar.

It would be desirable to drop the conditions that $q(t)$ and $a(t)$ be bounded away from zero (see Refs. [7] and [8]). By placing a stronger condition on $r(t)$ not only are we able to do this, but we also include the Z-type solutions which was not done in Refs. [7] or [8].

THEOREM 8. *Suppose conditions* (7)–(11) *hold and*

$$\int_{t_0}^{\infty} \left(\int_{t_0}^{w} [1/a(s)]\, ds\right) |\, r(w)|\, dw < \infty.$$

If $x(t)$ is a nonoscillatory or Z-type solution of Eq. (1), *then* $\lim_{t\to\infty} x(t) = 0$.

Proof. If $x(t)$ is a nonoscillatory or Z-type solution of Eq. (1), then $x(t)$ is either nonnegative or nonpositive for $t \geqslant t_1 \geqslant t_0$. From the first half of the proof of Theorem 7, it is clear that our hypotheses here imply that

$$\liminf_{t\to\infty} |\, x(t)| = 0.$$

Hence, if $x(t)$ is eventually monotonic, the conclusion of the theorem follows. We assume, then, that $x(t)$ is not eventually monotonic and that $x(t) \geqslant 0$ for $t \geqslant t_1$. An argument similar to the following one will hold in case $x(t)$ is nonpositive for $t \geqslant t_1$.

If the theorem does not hold, then there exists $A > 0$ such that for any $t_2 > t_1$ there exists $t_3 > t_2$ with $x(t_3) \geqslant A$. Choose $t_2 > t_1$ so that

$$\int_{t_2}^{\infty} \left(\int_{t_2}^{w} [1/a(s)]\, ds\right) |\, r(w)|\, dw < A/2,$$

and choose $t_3 > t_2$ such that $x'(t_3) = 0$ and $x(t_3) \geqslant A$. From Eq. (1) we have

$$x'(t) = (1/a(t)) \int_{t_3}^{t} [r(s) - q(s) f(x(s))\, g(x'(s))]\, ds.$$

Integrating again and then by parts we obtain

$$\begin{aligned} x(t) &= x(t_3) + \int_{t_3}^{t} \left\{(1/a(w)) \int_{t_3}^{w} [r(s) - q(s) f(x(s))\, g(x'(s))]\, ds\right\} dw \\ &= x(t_3) + \left(\int_{t_3}^{t} [1/a(s)]\, ds\right) \int_{t_3}^{t} [r(s) - q(s) f(x(s))\, g(x'(s))]\, ds \\ &\quad + \int_{t_3}^{t} \left(\int_{t_3}^{w} [1/a(s)]\, ds\right) [q(w) f(x(w))\, g(x'(w)) - r(w)]\, dw. \end{aligned}$$

Now if $t_4 > t_3$ is any zero of $x'(t)$, then we can use our above expression for $x'(t)$ to obtain that the first integral on the right hand side vanishes and thus

$$\begin{aligned} x(t_4) &\geqslant x(t_3) - \int_{t_3}^{t_4} \left(\int_{t_3}^{w} [1/a(s)]\, ds \right) r(w)\, dw \\ &\geqslant x(t_3) - \int_{t_3}^{\infty} \left(\int_{t_3}^{w} [1/a(s)]\, ds \right) | r(w)|\, dw \geqslant A/2. \end{aligned}$$

Hence $x(t)$ is bounded below by $A/2$ at every zero of $x'(t)$ for $t \geqslant t_3$. Since $x(t)$ is not ultimately monotonic, it follows that $x(t)$ is bounded below by $A/2$ for $t \geqslant t_3$ contradicting the fact that $\liminf_{t\to\infty} x(t) = 0$.

Next we prove two lemmas which are somewhat interesting in their own right. They will be used in the remainder of this paper.

LEMMA 9. *If* $r(t) \equiv 0$ *and conditions* (7)–(11) *hold, then all solutions of* (1) *are oscillatory or Z-type.*

Proof. Suppose the lemma if false. Then there is a solution $x(t)$ of Eq. (1) and $t_1 \geqslant t_0$ such that $x(t) \neq 0$ for $t \geqslant t_1$. From Eq. (1) we have

$$(a(t)\, x'(t))'/f(x(t)) + q(t)\, g(x'(t)) = 0,$$

and integrating, this becomes

$$\begin{aligned} -\int_{t_1}^{t} q(s)\, g(x'(s))\, ds = \int_{t_1}^{t} &\{a(s)\, f'(x(s))\, [x'(s)]^2/f^2(x(s))\}\, ds \\ &+ a(t)\, x'(t)/f(x(t)) - a(t_1)\, x'(t_1)/f(x(t_1)). \end{aligned}$$

Conditions (7) and (10) imply that the integral on the left approaches $+\infty$ as $t \to \infty$, and (9) implies that the integral on the right is nonnegative. Hence $a(t)\, x'(t)/f(x(t)) \to -\infty$ as $t \to \infty$. If $x(t) > 0$ for $t \geqslant t_1$, then there exists $t_2 \geqslant t_1$ such that $a(t)\, x'(t) < 0$ for $t \geqslant t_2$. Also, from (1) we have $(a(t)\, x'(t))' = -q(t)\, f(x(t))\, g(x'(t)) < 0$ for $t \geqslant t_2$, so $a(t)\, x'(t) - a(t_2)\, x'(t_2) < 0$ for $t \geqslant t_2$. Thus $x(t) < x(t_2) + a(t_2)\, x'(t_2) \int_{t_2}^{t} [1/a(s)]\, ds \to -\infty$ as $t \to \infty$ yielding a contradiction to $x(t) > 0$ for $t \geqslant t_2$. A similar proof holds if $x(t) < 0$ for $t \geqslant t_1$.

Remark. Wong [17] proved Lemma 9 for the case $a(t) \equiv 1$ without requiring condition (10).

LEMMA 10. *If* (7)–(11) *hold,* $\lim_{t\to\infty} H(t) = 0$, *and* $x(t)$ *is a nonoscillatory solution of Eq.* (1), *then* $\liminf_{t\to\infty} | x(t)| = 0$.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1); then $x(t) \neq 0$ for

$t \geqslant t_1 \geqslant t_0$, say $x(t) > 0$ for $t \geqslant t_1$. First observe that $x(t)$ is also a nonoscillatory solution of

$$(a(t)\,x'(t))' + [q(t) - r(t)/f(x(t))\,g(x'(t))]\,f(x(t))\,g(x'(t)) = 0$$

on $[t_1, \infty)$. If there exists $A > 0$ such that $x(t) \geqslant A$ for $t \geqslant t_1$, then $f(x(t)) \geqslant f(A) > 0$ for $t \geqslant t_1$. Thus since $\lim_{t\to\infty} H(t) = 0$, there exists $t_2 \geqslant t_1$ such that $H(t)/f(x(t))\,g(x'(t)) < \frac{1}{2}$ for $t \geqslant t_2$. This implies that

$$\int_{t_2}^{t} [q(s) - r(s)/f(x(s))\,g(x'(s))]\,ds$$

$$= \int_{t_2}^{t} q(s)\,[1 - H(s)/f(x(s))\,g(x'(s))]\,ds \geqslant (1/2)\int_{t_2}^{t} q(s)\,ds$$

so

$$\int_{t_2}^{\infty} [q(s) - r(s)/f(x(s))\,g(x'(s))]\,ds = \infty,$$

and by Lemma 9 we would have that $x(t)$ is oscillatory. A similar argument holds if $x(t) < 0$ for $t \geqslant t_1$.

Remark. Lemma 10 partially answers a question raised by Kartsatos [12] for nth order equations.

THEOREM 11. *Suppose conditions* (5)–(6) *and* (8)–(10) *hold,*

$$\lim_{t\to\infty} H(t) = 0, \qquad \int_{t_0}^{\infty} [|\,r(s)|/q(s)]\,ds < \infty$$

and there is a positive constant N such that

$$y^2/g(y) < N \text{ for all } y. \tag{12}$$

Then all solutions of Eq. (1) *are bounded. Also, if $x(t)$ is a nonoscillatory or Z-type solution of Eq.* (1), *then* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $V(x, y, t) = b(t)\,p(t)[F(x)/a(t) + G(y)/q(t)]$; then

$$V' \leqslant b(t)\,p(t)[y^2 a'(t)_-/g(y)\,q(t)\,a(t) + r(t)\,y/g(y)\,q(t)\,a(t)].$$

Conditions (5) and (6) imply that $p(t) \geqslant p_1 > 0$, $q(t) \geqslant q_1 > 0$, $b(t) \geqslant b_1 > 0$ and $a(t) \geqslant a_1 > 0$. Thus for any $t_1 \geqslant t_0$ an integration of V' yields

$$V(t) \leqslant V(t_1) + (N/q_1)\int_{t_1}^{t} [a'(s)_-/a(s)]\,ds$$

$$+ (1/a_1)\int_{t_1}^{t} [|\,r(s)|\;|\,y(s)|/g(y(s))\,q(s)]\,ds.$$

Now (12) implies that $|y|/g(y) < N_1$ for some $N_1 > 0$. Hence

$$F(x(t)) \leqslant a_2F(x(t_1))/a_1b_1p_1 + a_2G(y(t_1))/b_1p_1q_1$$
$$+ (a_2N/b_1p_1q_1) \int_{t_1}^{t} [a'(s)_-/a(s)]\, ds$$
$$+ (a_2N_1/a_1b_1p_1) \int_{t_1}^{t} [|r(s)|/q(s)]\, ds.$$

Thus $F(x(t))$ is bounded for $t \geqslant t_0$ and since (8) and (9) imply that $F(x) \to \infty$ as $|x| \to \infty$, we have that $x(t)$ is bounded.

If $x(t)$ is a nonoscillatory or Z-type solution of Eq. (1), then $\liminf_{t\to\infty} |x(t)| = 0$ by Lemma 10. If $x(t)$ is ultimately monotonic, then $\lim_{t\to\infty} x(t) = 0$. If $x(t)$ is not ultimately monotonic and $\epsilon > 0$ is given, choose $t_1 \geqslant t_0$ such that

$$y(t_1) = 0,\ F(x(t_1)) < a_1b_1p_1\epsilon/3a_2\,,\ \int_{t_1}^{\infty} [a'(s)_-/a(s)]\, ds < b_1p_1q_1\epsilon/3a_2N,$$

and

$$\int_{t_1}^{t} [|r(s)|/q(s)]\, ds < a_1b_1p_1\epsilon/3a_2N_1\,.$$

Then $F(x(t)) < \epsilon$ for $t \geqslant t_1$ which implies that $\lim_{t\to\infty} x(t) = 0$.

Remark. In Theorem 11, as well as in the following two theorems, the requirement that $H(t) \to 0$ as $t \to \infty$ is not needed to obtain boundedness of solutions. In this regard, condition (10) is needed in Theorem 13 but not in Theorem 11 or 12. Also, conditions (8) and (9) can be replaced by asking instead that $F(x) \to \infty$ as $|x| \to \infty$.

In the next theorem we relax condition (12) on $g(y)$ by imposing a stronger condition on $a(t)$.

THEOREM 12. *Assume that conditions* (5) *and* (8)–(10) *hold,*

$$a'(t) \geqslant 0,\ a(t) \leqslant a_2\,,\ \lim_{t\to\infty} H(t) = 0,\ \int_{t_0}^{\infty} [|r(s)|/q(s)]\, ds < \infty,$$

and there exists $L > 0$ *such that*

$$|y|/g(y) < L \text{ for all } y. \tag{13}$$

Then all solutions of Eq. (1) *are bounded, and if* $x(t)$ *is a nonoscillatory or* Z*-type solution of Eq.* (1), *we have* $\lim_{t\to\infty} x(t) = 0$.

Proof. Define $V(x, y, t) = p(t)[F(x)/a(t) + G(y)/q(t)]$; then

$$V' \leqslant p(t)\, r(t)\, y/g(y)\, q(t)\, a(t).$$

Now $a(t) \geqslant a_1 > 0$ since $a'(t) \geqslant 0$, so

$$V(t) \leqslant V(t_1) + (L/a_1) \int_{t_1}^{t} [| r(s)|/q(s)]\, ds$$

for $t \geqslant t_1 \geqslant t_0$. As in the proof of the previous theorem, it follows that $x(t)$ is bounded. If $x(t)$ is a nonoscillatory or Z-type solution, then

$$\liminf_{t\to\infty} | x(t)| = 0.$$

For a given $\epsilon > 0$ choose $t_1 \geqslant t_0$ so that $y(t_1) = 0, F(x(t_1)) < a_1 p_1 \epsilon/2a_2$, and

$$\int_{t_1}^{\infty} [| r(s)|/q(s)]\, ds < a_1 p_1 \epsilon/2a_2 L.$$

From this we again have $\lim_{t\to\infty} x(t) = 0$.

By imposing a stronger condition on $H(t)$ we can relax the conditions on $a(t)$ and $g(y)$ in Theorem 12.

THEOREM 13. *Suppose conditions* (4)–(6) *and* (8)–(10) *hold,*

$$\lim_{t\to\infty} H(t) = 0,$$

and

$$\int_{t_0}^{\infty} [| r(s)|/(q(s))^{1/2}]\, ds < \infty.$$

Then all solutions of Eq. (1) *are bounded. In addition, if $x(t)$ is a nonoscillatory or Z-type solution of Eq.* (1), *then* $\lim_{t\to\infty} x(t) = 0$.

Proof. Defining V as in the proof of Theorem 11 and differentiating and integrating we have

$$b(t)\, p(t)\, G(y(t))/q(t)$$
$$\leqslant V(t_1) + \int_{t_1}^{t} \{b(s)\, p(s)\, [y^2(s)\, a'(s)_- + | r(s)|\, | y(s)|]/g(y(s))\, q(s)\, a(s)\}\, ds$$

for $t \geqslant t_1 \geqslant t_0$. If $| y |/(q(t))^{1/2} \geqslant 1$, then $| y |/(q(t))^{1/2} \leqslant y^2/q(t) + 1$, and if $|y|/(q(t))^{1/2} \leqslant 1$, then $|y|/(q(t))^{1/2} \leqslant 1 + y^2/q(t)$. Now for $|y| \leqslant k$, $y^2/g(y) \leqslant D$ for some $D > 0$, so $y^2/g(y) \leqslant D + MG(y)$ for all y. Thus

$$\begin{aligned} &b(t)\, p(t)\, y^2(t)/g(y(t))\, q(t) \\ &\quad \leqslant b(t)\, p(t)\, D/q(t) + MV(t_1) + M \int_{t_1}^{t} \{b(s)\, p(s)\, [y^2(s)\, a'(s)_-/g(y(s))\, q(s) a(s) \\ &\qquad + |r(s)|\, y^2(s)/g(y(s))\, (q(s))^{3/2}\, a(s)]\}\, ds \\ &\qquad + M \int_{t_1}^{t} [b(s)\, p(s)\, |r(s)|/g(y(s))\, (q(s))^{1/2}\, a(s)]\, ds. \end{aligned}$$

Since

$$\begin{aligned} &M \int_{t_1}^{t} [b(s)\, p(s)\, |r(s)|/g(y(s))\, (q(s))^{1/2}\, a(s)]\, ds \\ &\qquad \leqslant (m/ca_1) \int_{t_1}^{\infty} [|r(s)|/(q(s))^{1/2}]\, ds \leqslant K_1 < \infty, \end{aligned}$$

we have

$$\begin{aligned} &b(t)\, p(t)\, y^2(t)/g(y(t))\, q(t) \\ &\quad \leqslant K_2 + M \int_{t_1}^{t} \{b(s)\, p(s)\, [y^2(s)\, a'(s)_-/g(y(s))\, q(s)\, a(s) \\ &\qquad + |r(s)|\, y^2(s)/g(y(s))\, (q(s))^{3/2} a_1]\}\, ds. \end{aligned}$$

Hence

$$\begin{aligned} &b(t)\, p(t)\, y^2(t)/g(y(t))\, q(t) \\ &\quad \leqslant K_2 \exp\left(M \int_{t_1}^{t} [a'(s)_-/a(s) + |r(s)|/a_1(q(s))^{1/2}]\, ds\right) \leqslant K_3 < \infty \end{aligned}$$

for all $t \geqslant t_1 \geqslant t_0$. From this we immediately obtain that $V(t)$ is bounded and the remainder of the proof proceeds as before.

At this point we might ask whether or not we could replace (5) by

$$\int_{t_0}^{\infty} [q'(s)_+/q(s)]\, ds < \infty$$

and obtain additional results. This condition bounds $q(t)$ from above and so asking that

$$\int_{t_0}^{\infty} [|r(s)|/q(s)]\, ds < \infty \qquad \text{or} \qquad \int_{t_0}^{\infty} [|r(s)|/(q(s))^{1/2}]\, ds < \infty$$

implies that

$$\int_{t_0}^{\infty} | r(s)| \, ds < \infty.$$

It turns out that even if we replace (5) in this manner, we would still need to bound $q(t)$ from below. Thus any results on the nonoscillatory or Z-type solutions of Eq. (1) that are obtainable in this manner would already be contained in Theorem 7.

We conclude this paper with two theorems on the behavior of the oscillatory solutions of Eq. (1). By combining these theorems with the preceding ones we could obtain results which would guarantee that all solutions of Eq. (1) tend to zero as $t \to \infty$. We leave the formulation of such results to the reader.

THEOREM 14. *Suppose conditions* (5), (8), *and* (13) *hold,*

$$a'(t) \geqslant 0, \; a(t) \leqslant a_2 \, , \; \lim_{t \to \infty} q(t) = \infty, \; \int_{t_0}^{\infty} [| r(s)|/q(s)] \, ds < \infty,$$

and

$$\int_{0}^{\pm\infty} [s/g(s)] \, ds < \infty.$$

If $x(t)$ is an oscillatory or Z-type solution of Eq. (1), *then* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be an oscillatory or Z-type solution of Eq. (1) and let $\epsilon > 0$ be given. Choose $t_1 \geqslant t_0$ such that $x(t_1) = 0$, $G(y)/q(t) < p_1\epsilon/2a_2$ for $t \geqslant t_1$, and

$$\int_{t_1}^{\infty} [| r(s)|/q(s)] \, ds < a_1 p_1 \epsilon/2a_2 L.$$

Then defining V as in the proof of Theorem 12, differentiating, and integrating, we have

$$p(t) F(x(t))/a(t) \leqslant V(t_1) + (L/a_1) \int_{t_1}^{t} [| r(s)|/q(s)] \, ds \qquad \text{for} \quad t \geqslant t_1 \, .$$

The conclusion of the theorem follows immediately.

THEOREM 15. *Assume that conditions* (5)–(6), (8) *and* (12) *hold,*

$$\lim_{t \to \infty} q(t) = \infty, \; \int_{t_0}^{\infty} [| r(s)|/q(s)] \, ds < \infty, \quad \textit{and} \quad \int_{0}^{\pm\infty} [s/g(s)] \, ds < \infty.$$

If $x(t)$ is an oscillatory or Z-type solution of Eq. (1), *then* $\lim_{t\to\infty} x(t) = 0$.

Proof. From the proof of Theorem 11 we have

$$F(x(t)) \leqslant a_2F(x(t_1))/a_1b_1p_1 + a_2G(y(t_1))/b_1p_1q(t_1)$$
$$+ (a_2N/b_1p_1q_1)\int_{t_1}^{t}[a'(s)_-/a(s)]\,ds + (a_2N_1/a_1b_1p_1)\int_{t_1}^{t}[|r(s)|/q(s)]\,ds.$$

Since

$$\int_0^{\pm\infty}[s/g(s)]\,ds < \infty,$$

$G(y) \leqslant K$ for all y. If $x(t)$ is an oscillatory or Z-type solution of Eq. (1) and $\epsilon > 0$ is given, choose $t_1 \geqslant t_0$ so that $x(t_1) = 0$, $a_2K/b_1p_1q(t) < \epsilon/3$ for $t \geqslant t_1$ and appropriate conditions on the above integrals are satisfied.

Remark. Notice that the conditions imposed on $g(y)$ in Theorems 14 and 15 cannot be relaxed since the equation

$$x'' + 4t^2x = 4 + 6/t^4 + 2\cos t^2$$

has the oscillatory solution $x(t) = \sin t^2 + 1/t^2$, $t \geqslant 1$, and this solution does not converge to zero as $t \to \infty$. Here $g(x') \equiv 1$.

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